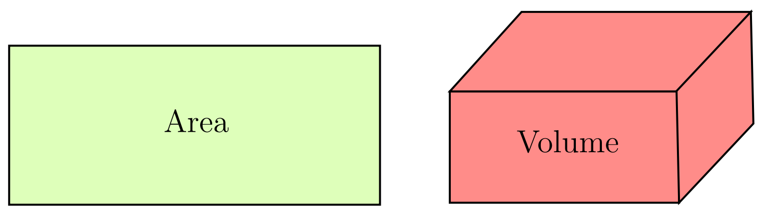
Grade 6 Math Circles
October 3/4/5, 2023
Area, Volume, and Optimization

Area and Volume

Area is the measure of the two dimensional space an object occupies. Up to this point, you've likely seen the area of squares and rectangles.

Volume is the measure of the three dimensional space an object occupies. You may have seen the volume rectangular prisms in school before.



Today, we will not only calculate the area and volume of some common shapes, but will also learn what it means to optimize the area and volume of these shapes.

Stop and Think

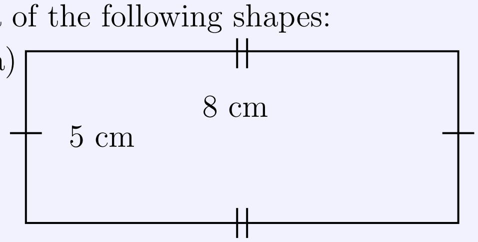
In which real life instances could we use area and volume?

Area of a Rectangle

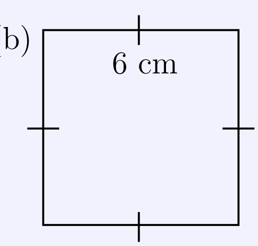
Exercise 1

Calculate the area of the following shapes:

(a)



(b)



Optimization of the Area of a Rectangle

Optimization is a topic in math that deals with maximizing or minimizing certain parts of a problem.

For example, the directions feature on Google Maps **optimizes** the path to your destination. It is designed to find the path that will take the minimum length of time to arrive. This type of optimization problem is known as a shortest path problem. Today, we won't be looking at a problem quite as challenging as this, but one that is still very practical.

We are interested in finding the maximum area of a rectangle with a given perimeter (and will later find the maximum area of other shapes).

Optimization Activity 1

Draw rectangles with **different** dimensions but the **same** perimeter. Use a perimeter of 20 cm. Calculate the area of each rectangle. Note which dimensions produced the largest area.

Hopefully you notice the rectangle with maximum area is actually a square! Why do you get a square? Let's look at the algebra behind the shape.

Let l = the length of the rectangle. Let w = the width of the rectangle.

Area: $A = lw$ and Perimeter: $P = 2l + 2w = 20$

We can write w in terms of l by doing some algebraic manipulations.

$$\begin{aligned} 2l + 2w &= 20 && \text{divide both sides by 2} \\ l + w &= 10 && \text{subtract } l \text{ from both sides} \\ w &= 10 - l && \text{so } w \text{ is the same as } 10 - l \end{aligned}$$

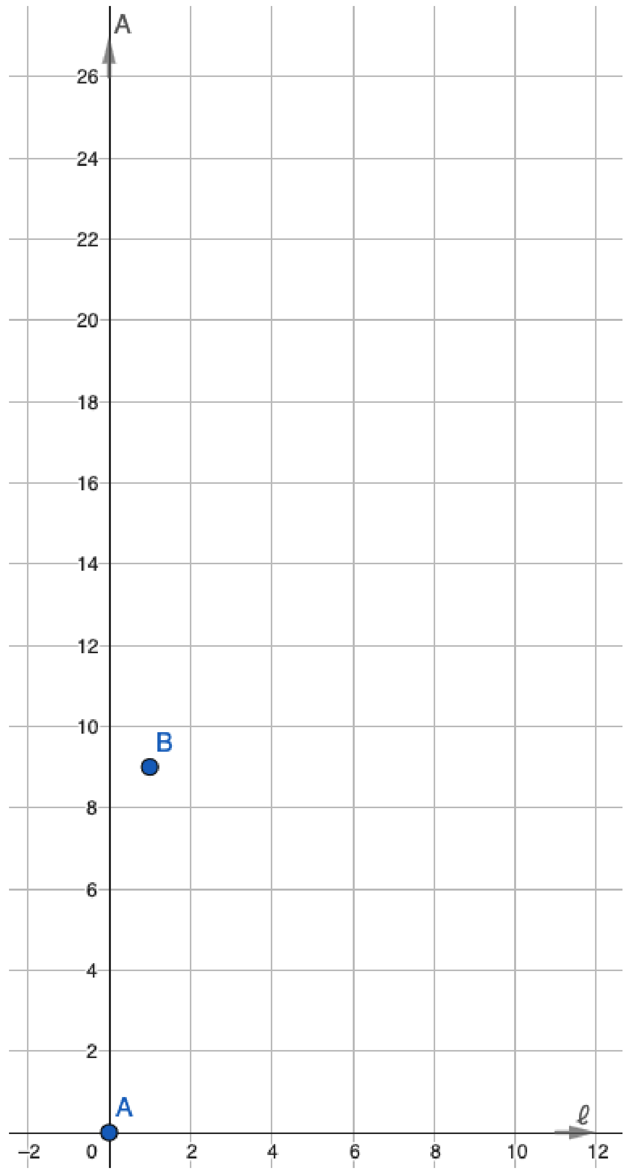
$$\begin{aligned} A &= lw && \text{replace } w \text{ with } 10 - l \\ A &= l(10 - l) \end{aligned}$$

We want to determine when the area is maximized. This is the same as asking, what does l need to be so that A is the largest it can be.

Let's calculate A for different values of l and record this information in the table below. Then plot the points on the graph. The first two are done for you.

After all the points are plotted, connect them with a smooth curve.

l	$A = l(10 - l)$
0 cm	0 cm^2
1 cm	9 cm^2
2 cm	
3 cm	
4 cm	
5 cm	
6 cm	
7 cm	
8 cm	
9 cm	
10 cm	



Stop and Think

When $l = 0$, $A = 0$. Does this make sense geometrically?

At which l value is A the largest?

Notice in both the table of values and the graph, the area is largest when $l = 5$ cm. The largest area is 25 cm^2.

Remember the width is $10 - l$. Therefore, when $l = 5$, $w = 5$ so the optimal area of a rectangle is a square!

GeoGebra Activity

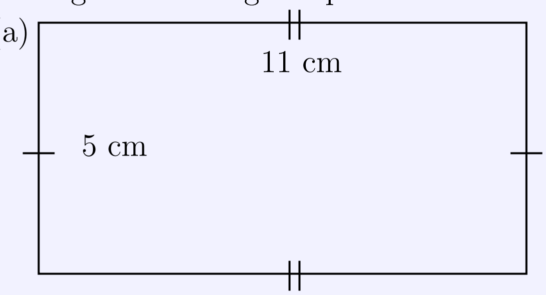
The following activity is designed to give a visual understanding of how area changes as the side lengths change.

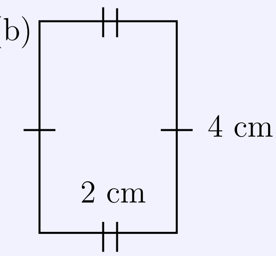
Click the following link to use the online graph and slider: https://www.geogebra.org/m/qcfhrggm. Use the slider to change the dimensions of the rectangle. Note the perimeter is always 20 cm.

NOTE: There are many other ways besides looking at a graph to solve this problem. One way is using *calculus*! This will appear much further in school.

Exercise 2

For each of the following rectangles, determine the dimensions that optimize the area of the rectangle with the given perimeter. Then determine the maximum area.

(a)


(b)


(c) A rectangle with a perimeter of 52 m

Area of Triangles and Parallelograms

To discover the areas of triangles and parallelograms, complete the following activities!

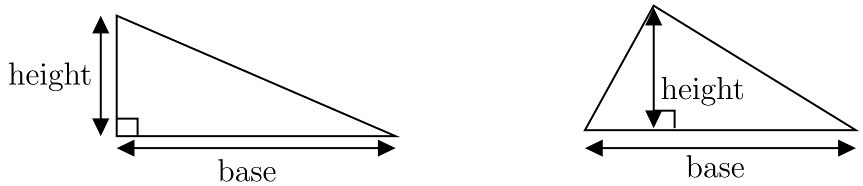
Paper Triangle Activity 1

You should have four pairs of **congruent** triangles. Congruent triangles have the same size and shape. (See page 14 for these triangles).

Orient your pairs of triangles labelled 1 and 2 so they form two rectangles. We know the formula for the area of a rectangle. What do you think the formula for the area of a triangle will be?

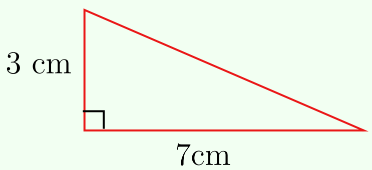
Since two congruent right-angled triangles can form a rectangle, this suggests the area of a triangle is **half** the area of a rectangle!

Area of a Triangle Formula: $A = \dfrac{(\text{base})\times(\text{height})}{2} = \dfrac{(b)(h)}{2}$



Example 1
Calculate the area of the following triangles.

a)



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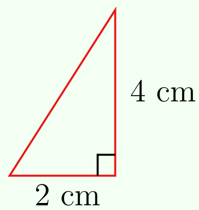
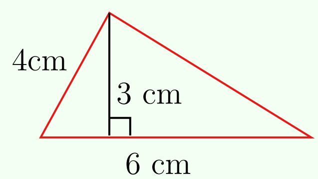


c)



Solution:

base = 7 cm, height = 3 cm
$$A = \frac{(7)(3)}{2}$$
$$A = 10.5 \text{ cm}^2$$

base = 2 cm, height = 4 cm
$$A = \frac{(2)(4)}{2}$$
$$A = 4 \text{ cm}^2$$

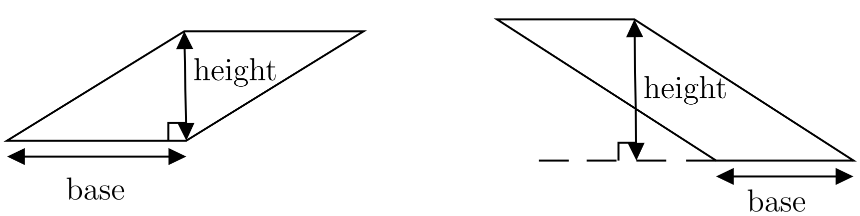
base = 6 cm, height = 3 cm
$$A = \frac{(6)(3)}{2}$$
$$A = 9 \text{ cm}^2$$

Paper Triangle Activity 2

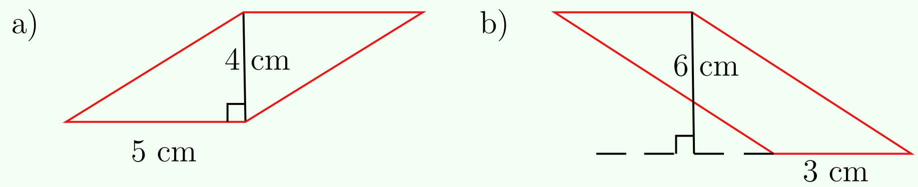
Using the pairs of triangles labelled 1 to 4 (from page 14), form four parallelograms. Try to predict the formula for the area of a parallelogram.

You can form parallelograms with two triangles, so the area of a parallelogram is the same as twice the area of a triangle!

Area of a Parallelogram Formula: $A = (\text{base})\times(\text{height}) = (b)(h)$



Example 2
Calculate the area of the following parallelograms.



Solution:

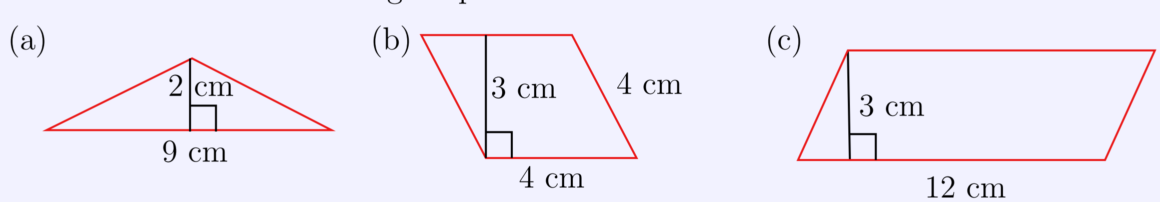
base = 5 cm, height = 4 cm
$$A = (5)(4)$$
$$A = 20 \text{ cm}^2$$

base = 3 cm, height = 6 cm
$$A = (3)(6)$$
$$A = 18 \text{ cm}^2$$

You may recall from previous knowledge that the rectangle is actually just a special case of the parallelogram. A rectangle is a parallelogram where each angle is equal to 90 degrees. It is for this reason that you can use the area of a parallelogram formula for rectangles too. This way, you have one less formula to remember!

Stop and Think
Can you make rectangles from the pairs of acute and obtuse triangles? If you are unsure, give it a try!

Exercise 3
Calculate the area of the following shapes.



Optimization of the Area of a Triangle

Now we want to optimize the area of a triangle when given the lengths of two of the sides.

Optimization Activity 2

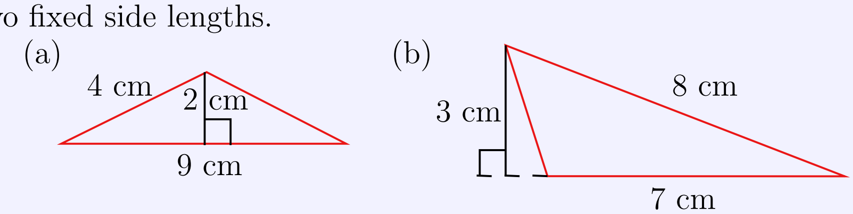
Visit the following link which leads to a graph and slider activity: https://www.geogebra.org/m/cy2v9qgc

What do you notice about the area as you change the angle between the sides? What angle gives the maximum area?

Notice the area is largest when the angle between the given sides is 90 degrees. This should make sense visually because when the angle is 90 degrees, the triangle's height is the greatest. Remember $A = \dfrac{(\text{base})\times(\text{height})}{2}$ for a triangle so in this case, maximizing area is equivalent to maximizing the height.

Exercise 4

First determine the current area of each triangle. Then determine the maximum area of each triangle with two fixed side lengths.



Stop and Think

Thinking about what we just discovered with the optimal angle for area of a triangle, predict how you maximize the area of a parallelogram given the lengths of two sides.

Applications of Area Optimization

Have you ever seen a floor plan of a house? Using a floor plan to fit as much furniture in a room as possible is an example of area optimization!

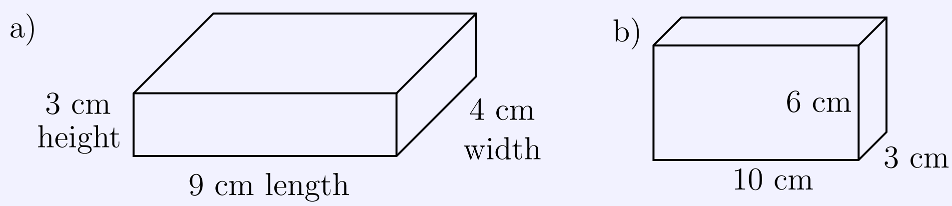
Volume of a Rectangular Prism

Volume is the three-dimensional analog of area. Let's start by calculating the volume of some rectangular prisms.

Recall: For a rectangular prism, $V = (\text{length})\times(\text{width})\times(\text{height}) = lwh$.

Exercise 5

Calculate the volume of each rectangular prism below.



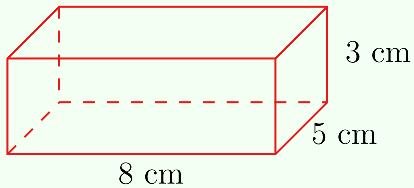
Optimization of the Volume of a Rectangular Prism

Now our goal is to maximize the volume of a rectangular prism with a given **surface area**. Surface area is the sum of the areas of all the faces of a 3D object.

Before we optimize, let's do some surface area practice. Remember that there are six faces on every rectangular prism. Calculate the area of each face and then add them together.

Example 3

Calculate the surface area of the following rectangular prism.



Solution:

There are six faces, but faces on opposite sides of the prism are congruent so they have the same area. Therefore, we can calculate the surface area of the three faces that meet at a vertex and then multiply by 2.

$SA = 2[(\text{length})(\text{width}) + (\text{length})(\text{height}) + (\text{width})(\text{height})]$

$SA = 2[(8)(5) + (8)(3) + (5)(3)]$

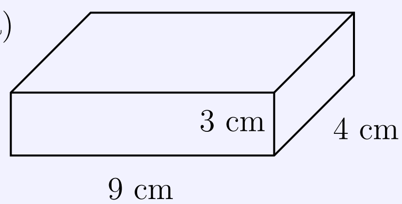
$SA = 2(79)$

$SA = 158 \text{ cm}^2$

Exercise 6

Calculate the surface area of the following rectangular prisms.

(a)



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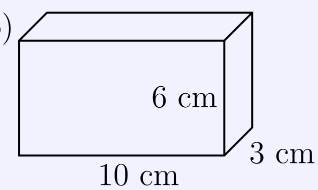


Stop and Think

Think back to optimizing the area of a rectangle. Remember that a square produced the maximum area. Using this fact, predict the dimensions of the rectangular prism that will produce the maximum volume.

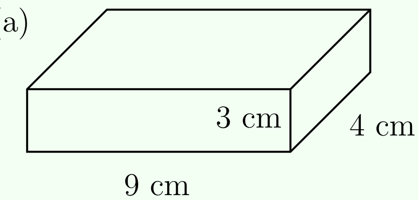
Use the following activity to see if your intuition is correct: https://www.geogebra.org/m/xn3ysugp. Note the surface area is always 96.

What dimensions produced the largest volume in the activity?

Example 4

The previous exercises had you find the volume and surface area of the following rectangular prism. Determine the dimensions that will maximize the volume while keeping surface area constant.

(a)

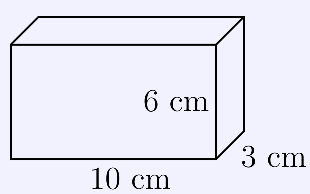


Solution:

The activity demonstrated that the cube is the optimal shape for the area of a rectangular prism. Therefore, we must find the side length of a cube with surface area equal to 150 cm^2.
There are 6 faces on a cube and each face is a square. Therefore, we need six squares to have a combined area of 150 cm^2.
$\frac{150}{6} = 25$, so the area of each square face is 25 cm^2.
Remember that the area of a square is its side length multiplied by itself (since the sides are equal). So we ask, "what number multiplied by itself is equal to 25?"
$5 \times 5 = 25$, so each edge of the cube is 5 cm and $V = 5 \times 5 \times 5 = 125$ cm^3.

Exercise 7

Maximize the volume of this rectangular prism while keeping the surface area constant.



Remember that to optimize a rectangular prism's volume with a given surface area, turn it into a **cube**!

Another Volume of a Rectangular Prism Formula

There's another way we can write the formula for the volume of a rectangular prism which is even more useful for discovering the volume formulas for other prisms.

Volume of a rectangular prism: $V =$ (area of rectangular base)$\times$(height).

The next section about the volume of a triangular prism should help you understand why this formula makes sense.

Volume of a Triangular Prism

The volume of a triangular prism is $V =$ (area of triangular base)$\times$(height). This is nearly identical to the volume of a rectangular prism!

Let's try to understand why this is the formula.

You can sort of think about a triangular prism as a triangle that has been "extended" into 3D space. So, if you start with the area of the base triangle, you can "extend" this area into 3D space to find the volume. This is why we multiply the base area by the height. The height is "how far we extend the base triangle."

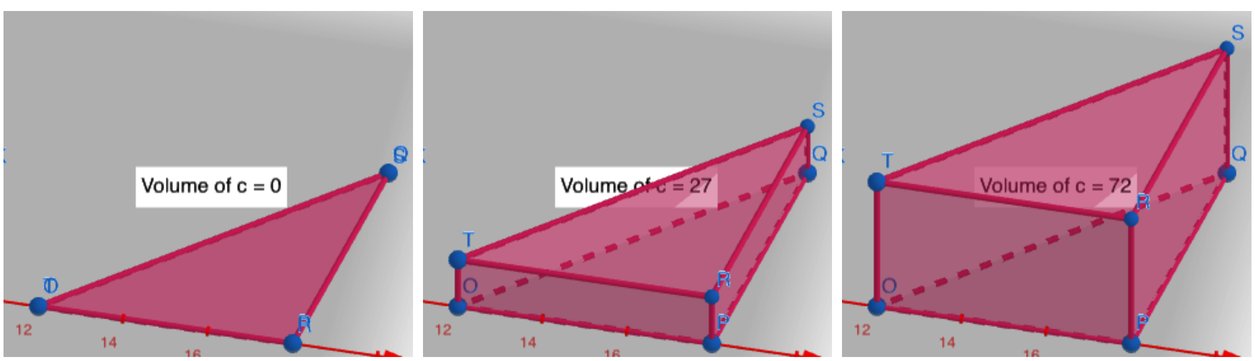


Figure 2: Extending a triangle's height to form a triangular prism.

Left image: Triangular prism of height 0 (so really it is just a triangle)

Middle image: Triangle prism of height 1.5

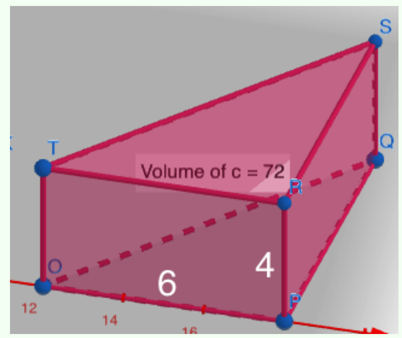
Right image: Triangular prism of height 4

Example 5

Look at the images in Figure 2. The length of the base triangle is 6 units and the heights of the prisms are 0, 1.5, and 4 units in the left, middle, and right images, respectively. The volumes are 0, 27, and 72 units^3. Find the height of the base triangle.

Solution:

Let's use the rightmost triangular prism for our solution. We need the area of the base triangle multiplied by the height of the prism to be equal to the volume.



We know the following:

- The height of the prism is 4 units.
- The volume of the prism is 72 units^3.
- The length of the base triangle is 6 units.

We need the following:

- The area of the base triangle such that (area of the base triangle)$\times 4 = 72$

Since $18 \times 4 = 72$, the area of the base triangle must be 18 units^2.

Now we need the $\dfrac{\text{(base of base triangle)} \times \text{(height of base triangle)}}{2} =$ area of base triangle.

The length of the base triangle is 6 units, so we need $\dfrac{6 \times \text{(height of base triangle)}}{2} = 18$.

Therefore, the height of the base triangle is 6 units.

What's great about this idea of starting with the base area and "extending" it to get the prism is this works for ALL prisms. As long as you know how to find the area of the base shape, you can easily find the volume by simply multiplying by the prism's height.

An Application of Volume Optimization

Has your family ever gone for a long vacation and to pack the car *very* strategically to fit everything? This is an example of volume optimization! You can pack your luggage on the two dimensional surface of the car, but can also stack items on top of each other. It's three dimensional!

Summary

Area Formulas

Area of a Rectangle: $A = (\text{length})\times(\text{width})$ or $A = (\text{base})\times(\text{height})$

Area of a Triangle: $A = \dfrac{(\text{base})\times(\text{height})}{2}$

Area of a Parallelogram: $A = (\text{base})\times(\text{height})$

Area Optimization

Shape	What we are Changing	What we are Given	Optimal Shape
Rectangle	Side Lengths	Perimeter	Square
Triangle	Angle Between Sides	Two Side Lengths	Right-Angled Triangle

Volume Formulas

Volume of a Rectangular Prism:

- $V = (\text{length})\times(\text{width})\times(\text{height})$ or;
- $V = (\text{area of rectangular base})\times(\text{height})$

Volume of a Triangular Prism: $V = (\text{area of triangular base})\times(\text{height})$

Volume of Any Prism: $V = (\text{area of base})\times(\text{height})$

Volume Optimization

Shape	What we are Changing	What we are Given	Optimal Shape
Rectangular Prism	Three Side Lengths	Surface Area	Cube

Triangles for Activities